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Cauchy Boundaries of Space-Times

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The definition of space-time is reformulated so as to elucidate the role of topology and the so-called *dynamical structure of space-time.* The nonuniqueness of the Cauchy boundary for space-times is discussed. The definition of *space-time with Cauchy boundary* (*C-space-time*) is proposed and Cauchy boundaries of C-space times for Friedmanian models are constructed.

INTRODUCTION

During more than 70 years of relativistic physics, the smooth Lorentz manifold concept as a model for space-time has played a central role within the theoretical scheme of relativity (Beem and Ehrlich, 1981; Hawking and Ellis, 1973; Geroch and Horowitz, 1979). In spite of its formal similarity to the Riemannian case, the notion of the Lorentz manifold contains a richer set of different structures, such as conformal, projective, causal, and chronological structures, and each of these structures is supposed to have some important physical connotations. The richness of these structures, in contrast to the Riemannian case, "screens out" the topological structure. To be more precise, in the Lorentz case, the topological structure is not given in an explicit way and has no metric character. In my view, this is a great drawback of the Lorentz manifold definition: all topological considerations become difficult. In particular, attempts to define a space-time boundary as a "place" for singularities have met with serious difficulties, also because of these topological intricacies. For instance, in Schmidt's elegant construction (Schmidt, 1971) of the singular boundary of space-time (M, g^{-}) the topological methods are used not in the space-time itself, but in the bundle of orthonormal frames $O(M)$. The *Cauchy boundary* (Hawking and Ellis, 1973, p. 282) $\partial O(M) = \overline{O(M)} - O(M)$ [$\overline{O(M)}$ is the Cauchy completion of $O(M)$] "projected" down to M defines the singular boundary of

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space-time (b-boundary). Unfortunately, the construction leads to physically unacceptable non-Hausdorff phenomena in special cases of space-times with b-boundary (Bosshard, 1976; Johnson, 1977).

There is, however, a possibility to reformulate the definition of space-time (Section 1) so as to avoid the majority of the above problems. The new definition not only takes control over topology from the very beginning, but it also takes into account what I shall call the dynamical structure of space-time.

In Section 2 I discuss the nonuniqueness of the Cauchy boundary notion for space-times. Cauchy boundaries for Friedmanian models are constructed in Section 3. Section 4 contains a formal definition of space-time with the Cauchy boundary (C-space-time).

1. DYNAMICAL STRUCTURE OF SPACE-TIME

The reformulation of the definition of space-time is based on the commonly known theorem stating that a manifold (M, τ) (τ is topology on M) admits the existence of a Lorentz structure g^- if and only if (M, τ) is a paracompact manifold equipped with C^1-direction field DIR(M). As is well known, every C^1-manifold (M, τ) is paracompact if and only if it is a Riemannian manifold (M, g) (Steenrod, 1951; Geroch and Horowitz, 1979; Choqet-Bruhat *et al.*, 1974). Therefore, every Lorentz C^1-manifold (M, g^-, τ) is a Riemannian manifold, in the above sense, equipped with C^1-direction field, and consequently, as a topological space, it is the metric space with the metric topology τ_ρ given by the distance function $\rho_g(x, y)$, $x, y \in M$, defined as in the following way. The distance $\rho_g(x, y)$ between two points $x, y \in M$ is, by definition, the infimum of the lengths of all piecewise differentiable curves of class C^1 joining x and y. The topology τ_ρ is equivalent to the topology τ (see, for example, Beem and Ehrlich, 1981; Choqet-Bruhat *et al.*, 1974). Unfortunately, the correspondence $g^- \leftrightarrow (g, \mathrm{DIR}(M))$ is not one to one (Beem and Ehrlich, 1981; Geroch, 1971). Indeed, many different pairs $(g, \mathrm{DIR}(M))$ correspond to the same Lorentz tensor g^- on M. Every two such pairs will be called *g^--equivalent.* Up to this equivalence, the triple $(M, g, \mathrm{DIR}(M))$ defines the Lorentz manifold (M, g^-, τ).

Space-time is a time-orientable Lorentz manifold; therefore, the direction field DIR(M) is generated by a nowhere vanishing timelike vector field $\xi \in TM$. Vice versa, if DIR(M) is generated by such vector field ξ, the corresponding Lorentz manifold is time-orientable (Geroch and Horowitz, 1979; Godbillon, 1983). Moreover, if ξ generates DIR(M), then $\lambda\xi$ does this, too (λ is a nonvanishing scalar field on M). It can be shown that it is possible to choose λ in such a manner that the field $\lambda\xi$ be complete.

Therefore the set of integral curves of this field

$$\{\gamma_x: \mathbb{R} \to M: \dot{\gamma}_x = (\lambda\xi)(\gamma), \gamma_x(0) = x, x \in M\}$$

defines a smooth (C^1) dynamical system (M, ϕ_t) without critical points, such that $\phi_t(x) = \gamma_x(t)$ (Hirsch and Smale, 1974; Godbillon, 1983).

In such a way, with every time-orientable Lorentz manifold (M, g^-, τ) a triple (M, g, ϕ_t) can be associated (up to g^--equivalence). This allows one to formulate the following.

Definition 1. The g^--equivalence class of the triple $\langle M, g, \phi_t\rangle$ is said to be a space-time if:

1. (M, g) is a connected, Hausdorff, C^∞-Riemannian manifold,
2. (M, ϕ_t) is a smooth (C^1) dynamical system on M without critical points.

ϕ_t will be called the *dynamical structure of space-time.* $\langle\cdot\rangle$ denotes the g^--equivalence class; however, in practice we shall usually deal with representatives of such classes.

The above definition has a formal character. However, we can assume that connection between g^- and (g, ϕ_t) is given globally by the well-known formula (see Choqet-Bruhat *et al.*, 1974, p. 280)

$$g^-_{ab} = g_{ab} - 2\xi_a\xi_b / g_{cd}\xi^c\xi^d, \qquad a, b, c, d = 1, \ldots, \dim M \qquad (1)$$

which makes the definition useful in practice. The globally defined vector field ξ on M without critical points is determined by the dynamical structure ϕ_t. The Riemannian metric g defines the above-mentioned function ρ_g and consequently the metric topology τ_ρ (I shall always omit the letter g) equivalent to the topology τ of (M, g^-, τ).

The definition is equivalent to the usual definition of space-time. However, its complicated form is not useful for investigations of space-times themselves; it is suitable (in my view) for studying topological properties of space-times and for investigations of space-times with singularities.

2. CAUCHY BOUNDARY OF SPACE-TIME

In this section, I make some mathematical observations in order to elucidate the special character of the Cauchy boundary of space-time.

First, a space-time, according to Definition 1, is an equivalence class of dynamical systems on Riemannian manifolds. Thus, with a given space-time $\langle M, g, \phi_t\rangle$ we can associate a family $\mathcal{R}$ of Riemannian manifolds $(M, \tilde{g})$ for which there is $\tilde{\phi}_t$ such that $(M, \tilde{g}, \tilde{\phi}_t) \in \langle M, g, \phi_t\rangle$.

Second, there is the general conviction that points of a space-time boundary would be accumulation points of space-time events. Therefore, it is natural to assume that a singular boundary of space-time could be defined by means of Cauchy sequences on M.

In the case of Riemannian manifolds, on the strength of the Hopf–Rinow theorem (see for, e.g., (Kobayashi and Nomizu, 1963, p. 172; Beem and Ehrlich, 1981), the Cauchy incompleteness is equivalent to the geodesic incompleteness. Consequently, the Cauchy boundary $\partial M = \bar{M} - M$ ($\bar{M}$ is the Cauchy completion of M) and the geodesic boundary are the same.

Unfortunately, the theorem is not true for Lorentzian manifolds and the usefulness of the g-boundary and the Cauchy boundary for defining a singular boundary of space-time have to be analyzed separately.

Achievements and difficulties of the g-boundary are well known (see e.g., Beem and Ehrlich, 1981; Hawking and Ellis, 1973). In this paper I discuss the possibility of defining singular boundaries of space-times by means of Cauchy sequences on M.

A space-time $\langle M, g, \phi_t \rangle$, as a topological space, is metrizable. With the family $\mathscr{R}$ of Riemannian manifolds the family $\tilde{\mathscr{R}} = \{(M, \rho_g): (M, g) \in \mathscr{R}\}$ of metric spaces can be associated. Every two metric spaces $(M, \rho_1), (M, \rho_2) \in \tilde{\mathscr{R}}$ are topologically equivalent.

The notion of Cauchy sequence is not invariant under homeomorphic transformations of metric spaces; the sequence $\{x_n\} \subset M$ which is the Cauchy sequence for $(M, \rho_1) \in \tilde{\mathscr{R}}$ need not be one for $(M, \rho_2) \in \tilde{\mathscr{R}}$. However, the notion is invariant under uniformly homeomorphic transformations.

In the following, two Riemannian manifolds $(M, g_1), (M, g_2) \in \mathscr{R}$ are called *uniformly equivalent* ($\sim u$) if the corresponding metric spaces (M, ρ_1), $(M, \rho_2) \in \tilde{\mathscr{R}}$ are uniformly homeomorphic.

The relation $\sim u$ divides the family $\mathscr{R}$ into classes $\mathscr{R}/\sim u := \mathscr{R}^{\cdot}$. It is easy to prove that, for $[(M, g)] \in \mathscr{R}^{\cdot}$, the following statements are true.

1. If $(M, g_1), (M, g_2) \in [(M, g)]$, then the Cauchy completions $(\bar{M}, \bar{\rho}_i)$, $i = 1, 2$, of metric spaces (M, ρ_i) are uniformly homeomorphic; $\bar{M}_i$ denotes the Cauchy completions of M with respect to ρ_i.
2. The Cauchy boundaries ∂M_i, $i = 1, 2$, for every two representatives of the class $[(M, g)]$ have the same topological properties.

From the above argument it can be seen that, in general, a given space-time has no uniquely defined boundary by Cauchy sequences $\{x_n\} \subset M$.

In this situation the natural assumption of the uniqueness of the Cauchy boundary (as a singular boundary) of a given space-time, with respect to its topological properties, compels us to distinguish one class $[(M, g)]_0 \in \mathscr{R}^{\cdot}$. The geometrical construction of such a distinguished class $[(M, g)]_0$ and its connection with the b-boundary construction is discussed in Gruszczak *et al.* (to appear). In the present paper I shall only present some examples of cosmological models in which, among all classes $\mathscr{R}^{\cdot}$ for a given model, there exists a class $[(M, g)]_0$ such that its Cauchy boundary has topological properties one should expect from one's theoretical experience.

3. CAUCHY BOUNDARIES OF FRIEDMANIAN MODELS

Example 1. Let us consider the flat Friedmanian model with radiation:

$$ds^2 = R^2(x^0)[(-dx^0)^2 + \delta_{km}\, dx^k\, dx^m], \qquad R(x^0) = R_0 x^0, \qquad k, m = 1, 2, 3$$

In this case, the dynamical structure ϕ_t can be given by the vector field $\xi = (1, 0, 0, 0)$. The boundary of our model (see Definition 1)

$$(\mathbb{R}^4_+, g, \phi_t\colon \mathbb{R}^4_+ \ni x \to (x^0 e^t, x^k) \in \mathbb{R}^4_+), \qquad k = 1, 2, 3$$

where $g_{ab} = R^2(x^0)\delta_{ab}$, $a, b = 0, \ldots, 3$, is determined by formula (1), is the Cauchy boundary of the Riemannian manifold $(\mathbb{R}^4_+, g)$. It can be easily shown, by straightforward calculation, that in this case the Cauchy boundary (the cosmological singularity) is a single point.

Example 2. Let us consider the closed Friedmanian model with radiation (M, g, ϕ_t), where:

(a) M is the open area in $\mathbb{R}^4$ between two concentric three-spheres $S_1(0, \rho_1)$ and $S_2(0, \rho_2)$.

(b) g is given by the following line element written in four-dimensional spherical coordinates:

$$ds^2 = R^2(\rho)\{d\rho^2 + \rho^2[d\psi^2 + \sin^2\psi\, (d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)]\}$$

where $\rho \in (\rho_1, \rho_2)$, $\psi \in (0, \pi)$, $\vartheta \in (0, \pi)$, $\varphi \in (0, 2\pi)$, $\rho_1 = 1$, $\rho_2 = e^{\pi}$, and b = constant of the model, $R(\rho) = b\rho^{-1}\sin(\ln \rho)$.

(c) The dynamical structure ϕ_t is given by

$$\phi_t(\rho, \psi, \vartheta, \varphi) = (f(\rho, t), \psi, \vartheta, \varphi)$$

$$f(\rho, t) = [\rho_2(\rho - \rho_1)e^t + \rho_1(\rho_2 - \rho)e^{-t}][(\rho - \rho_1)e^t + (\rho_2 - \rho)e^{-t}]^{-1}$$

The Cauchy boundary of this model consists of the three-spheres $S_1(0, \rho_1)$ and $S_2(0, \rho_2)$ which are, topologically, two separated points. These points coincide with the initial and final cosmological singularities, respectively.

This model will be discussed in detail in a forthcoming paper. Both flat and closed cosmological models as space-times with boundaries are Hausdorff topological spaces.

This result, for the closed Friedmanian model, remains in contrast with the b-boundary of this model, and avoids all the troubles of the latter with non-Hausdorff behavior and identification of the initial and final singularities (Bosshard, 1976; Johnson, 1977).

4. SPACE-TIME WITH CAUCHY BOUNDARY

The above examples have shown that, for a given space-time, the dynamical structure ϕ_t can be chosen in such a way that the corresponding Cauchy boundary of (M, g) has physically reasonable topological properties This result encourages the following definition.

Definition 2. The triple $[M, g, \phi_t]_0$ will be called a *space-time with Cauchy boundary (C-space-time).*

The symbol $[M, g, \phi_t]_0$ denotes the set of all g^--equivalent triples $(M, \tilde{g}, \tilde{\phi}_t) \in \langle M, g, \phi_t \rangle$ such that corresponding Riemannian manifolds $(M, \tilde{g})$ are uniformly equivalent and form a class $[(M, g)]_0 \in \mathscr{R}^{\cdot}$. The class $[(M, g)]_0$ can be distinguished by means of physical arguments.

Naturally, the Cauchy boundary, for a given C-space-time, is defined uniquely and can be treated as its singular boundary.

Definition 3. The Cauchy boundary of every $(M, \tilde{g}) \in [(M, g)]_0 \in \mathscr{R}^{\cdot}$ can serve as a *singular boundary of the C-space-time* $[M, g, \phi_t]_0$.

The C-space-time notion differs from the usual space-time notion (Definition 1) by an additional assumption concerning the space-time's uniform structure. This assumption serves only to make the Cauchy boundary of a given space-time (M, g^-, τ) unique, and does not change its geometrical and topological properties; $[M, g, \phi_t]_0$ is a set of g^--equivalent triples $(M, \tilde{g}, \tilde{\phi}_t)$ which define the same space-time (M, g^-, τ). The assumption has a global character and it has no influence on all local space-time properties, so it is admissible in the framework of general relativity. The C-space-time concept is a workable concept which deserves further elaboration.

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